On a certain generalization of triangle singularities

Kenji Hashimoto, Hwayoung Lee, Kazushi Ueda

Abstract

Triangle singularities are Fuchsian singularities associated with von Dyck groups, which are index two subgroups of Schwarz triangle groups. Hypersurface triangle singularities are classified by Dolgachev, and give 14 exceptional unimodal singularities classified by Arnold. We introduce a generalization of triangle singularities to higher dimensions, show that there are only finitely many hypersurface singularities of this type in each dimension, and give a complete list in dimension 3.

1 Introduction

Let $\boldsymbol{p} = (p, q, r)$ be a triple of positive integers satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1. \tag{1.1}$$

The *Schwarz triangle group*

$$\Delta = \left\langle a, b, c \;\middle|\; a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = e \right\rangle \tag{1.2}$$

is the reflection group generated by reflections along edges of the hyperbolic triangle in the upper half plane $\mathbb{H}$ with angles π/p, π/q, and π/r. The *von Dyck group*

$$\Gamma = \langle x, y, z \mid x^p = y^q = z^r = xyz = e \rangle \tag{1.3}$$

is the subgroup of Δ of index two consisting of products of even numbers of reflections. It is a cocompact Fuchsian group, and the orbifold quotient $\mathbb{X} = [\mathbb{H}/\Gamma]$ has three orbifold points with stabilizers $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/q\mathbb{Z}$, and $\mathbb{Z}/r\mathbb{Z}$. The triple $\boldsymbol{p} = (p, q, r)$ is called the *signature* of the Fuchsian group Γ.

Smooth rational orbifolds are studied in detail by Geigle and Lenzing [GL87] under the name of *weighted projective lines*. In particular, the orbifold $\mathbb{X}$ can be described as

$$\mathbb{X} = [(\operatorname{Spec} T \setminus \mathbf{0})/K], \tag{1.4}$$

where

$$T = \mathbb{C}[X, Y, Z]/(X^p + Y^q + Z^r) = \bigoplus_{\vec{k} \in L} T_{\vec{k}} \tag{1.5}$$

is a two-dimensional ring graded by the abelian group

$$L = \mathbb{Z}\vec{X} \oplus \mathbb{Z}\vec{Y} \oplus \mathbb{Z}\vec{Z} \oplus \mathbb{Z}\vec{c}/(p\vec{X} - \vec{c}, q\vec{Y} - \vec{c}, r\vec{Z} - \vec{c}) \tag{1.6}$$

Signature	Generators	Weights	Relation
$(2,3,7)$	(X,Y,Z)	$(21,14,6;42)$	$x^2+y^3+z^7$
$(2,3,8)$	(XZ,Y,Z^2)	$(15,8,6;30)$	$x^2+y^3z+z^5$
$(2,3,9)$	(X,YZ,Z^3)	$(9,8,6;24)$	$x^2z+y^3+z^4$
$(2,4,5)$	(XY,Y^2,Z)	$(15,10,4;30)$	$x^2+y^3+yz^5$
$(2,4,6)$	(XYZ,Y^2,Z^2)	$(11,6,4;22)$	$x^2+y^3z+yz^4$
$(2,4,7)$	(XY,Y^2Z,Z^3)	$(7,6,4;18)$	$x^2z+y^3+yz^3$
$(2,5,5)$	(Y^5,X,YZ)	$(10,5,4;20)$	$x^2+xy^2+z^5$
$(2,5,6)$	(Y^4,XZ,YZ^2)	$(6,5,4;16)$	$xy^2+x^2z+z^4$
$(3,3,4)$	(X^3,XY,Z)	$(12,8,3;24)$	$x^2+xz^4+y^3$
$(3,3,5)$	(X^3Z,XY,Z^2)	$(9,5,3;18)$	$x^2+xz^3+y^3z$
$(3,3,6)$	(X^3,XYZ,Z^3)	$(6,5,3;15)$	$x^2z+xz^3+y^3$
$(3,4,4)$	(XY^4,X^2,YZ)	$(8,4,3;16)$	$x^2+xy^2+yz^4$
$(3,4,5)$	(XY^3,X^2Z,YZ^2)	$(5,4,3;13)$	$xy^2+x^2z+yz^3$
$(4,4,4)$	(X^4,Y^4,XYZ)	$(4,4,3;12)$	$x^2y+xy^2+z^4$

Table 1.1: Hypersurface triangle singularities

of rank one, which is the group of characters of the algebraic group

$$K = \operatorname{Spec} \mathbb{C}[L]. \tag{1.7}$$

Here the grading is given by $X \in T_{\vec{X}}$, etc. The *dualizing element* is defined by

$$\vec{\omega} = \vec{c} - \vec{X} - \vec{Y} - \vec{Z}, \tag{1.8}$$

and the canonical ring of the orbifold $\mathbb{X}$ is given by

$$R = \bigoplus_{k=0}^{\infty} R_k, \qquad R_k = T_{k\vec{\omega}}. \tag{1.9}$$

The isolated singularity at the origin of the scheme $\operatorname{Spec} R$ is called the *triangle singularity* with signature $\boldsymbol{p} = (p,q,r)$.

Let $\mathfrak{m} = \bigoplus_{k=1}^{\infty} R_k$ denote the irrelevant ideal of R. The dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$ is called the *embedding dimension* of R. It is known that the embedding dimension of R coincides with the minimum number of generators of R (see Lemma 2.1). A graded ring is said to be a *hypersurface* if the embedding dimension is greater than the Krull dimension by one.

Theorem 1.1 ([Dol75], cf. also [Mil75, Wag80])**.** *The ring R is a hypersurface if and only if the signature $\boldsymbol{p}$ is one of the 14 signatures shown in Table 1.1.*

Table 1.1 coincides with the list of weighted homogeneous exceptional unimodal singularities classified by Arnold [Arn75]. The signature in this context is called the *Dolgachev number* of the exceptional unimodal singularity. Besides the Dolgachev number, each exceptional unimodal singularity comes with another triple of positive integers called the *Gabrielov number*,

which describes the Milnor lattice of the singularity [Gab74]. This leads to the discovery of *strange duality* by Arnold, which states that exceptional unimodal singularities come in pairs in such a way that the Dolgachev number and the Gabrielov number are interchanged. Strange duality is described in terms of an exchange of the algebraic lattice and the transcendental lattice of a K3 surface by Dolgachev and Nikulin [Dol83, Nik79] and Pinkham [Pin77], and is now considered as a precursor of *mirror symmetry*.

In this paper, we consider the following generalization of triangle singularities. Let n be an integer greater than 3 and $\boldsymbol{p} = (p_1, \dots, p_n)$ be a sequence of integers called a *signature*. In what follows, we assume that $\boldsymbol{p}$ satisfies

$$\sum_{i=1}^{n} \frac{1}{p_i} < 1. \tag{1.10}$$

Consider the ring

$$T = \mathbb{C}[X_1, \dots, X_n] \Bigg/ \left(\sum_{i=1}^{n} X_i^{p_i} \right), \tag{1.11}$$

which is graded by the abelian group

$$L = \mathbb{Z}\vec{X}_1 \oplus \cdots \oplus \mathbb{Z}\vec{X}_n \oplus \mathbb{Z}\vec{c} \Big/ \left(p_i \vec{X}_i - \vec{c} \right) \tag{1.12}$$

of rank one. Here the grading is given by $X_i \in T_{\vec{X}_i}$ for all $i \in \{1, \dots, n\}$. The *dualizing element* is defined by

$$\vec{\omega} = \vec{c} - \sum_{i=1}^{n} \vec{X}_i, \tag{1.13}$$

and the *canonical ring* is defined by

$$R = \bigoplus_{k=0}^{\infty} R_k, \qquad R_k = T_{k\vec{\omega}}. \tag{1.14}$$

The singularity of $\operatorname{Spec} R$ is a higher-dimensional generalization of triangle singularities. The main result in this paper is the finiteness of hypersurface singularities of this type:

Theorem 1.2. *For any integer n greater than 3, there are only finitely many signatures $\boldsymbol{p} = (p_1, \dots, p_n)$ such that R is a hypersurface.*

Our proof of Theorem 1.2 gives an algorithm to classify all hypersurface generalized triangle singularities for any given $n \geq 4$. The list of hypersurface generalized triangle singularities for $n = 4$ is shown in Table 1.2.

Our proof of Theorem 1.2 also gives the following:

Theorem 1.3. *For any integer n greater than 3, the generalized triangle singularity associated with signature $\boldsymbol{p} = (p_1, \dots, p_n)$ is an isolated hypersurface singularity if and only if*

$$\sum_{i=1}^{n} \frac{1}{p_i} + \frac{1}{\prod_{i=1}^{n} p_i} = 1. \tag{1.15}$$

A slight generalization

$$\sum_{i=1}^{n} \frac{1}{p_i} + \frac{1}{\prod_{i=1}^{n} p_i} \in \mathbb{Z} \tag{1.16}$$

of the Diophantine equation (1.15) is equivalent to the *improper Znám problem*, which asks for a sequence $(p_1, \ldots, p_n)$ of integers satisfying $p_i \mid \prod_{j \in \{1,\ldots,n\} \setminus \{i\}} p_j + 1$ for all $i \in \{1, \ldots, n\}$. It is not known whether (1.16) implies (1.15).

The *a-invariant* $a = a(R) \in \mathbb{Z}$ of a graded Gorenstein ring R is defined by

$$K_R = R(a), \tag{1.17}$$

where K_R is the graded canonical module of R. If R is a hypersurface generated by n elements of degrees $a_1, \ldots, a_n$ with one relation of degree h, then the a-invariant of R is given by

$$a = h - a_1 - \cdots - a_n. \tag{1.18}$$

Theorem 1.4. *If R is a hypersurface generalized triangle singularity, then one has*

$$a(R) = 1. \tag{1.19}$$

The equality (1.19) also holds for hypersurface (ordinary) triangle singularities (cf. [Wag80, Proposition 2.8]). It follows from (1.19) that one can compactify $\operatorname{Spec} R$ to a weighted projective hypersurface with trivial canonical bundle by adding one variable of degree one. It is an interesting problem to study when this hypersurface admits a smoothing.

Acknowledgements: We thank Korea Institute for Advanced Study for financial support and excellent research environment. K. U. is supported by JSPS Grant-in-Aid for Young Scientists No. 24740043.

Signature	Generators	Weights	Relation
$(2,3,7,43)$	(X,Y,Z,W)	$(903,602,258,42;1806)$	$x^2+y^3+z^7+w^{43}$
$(2,3,7,44)$	(XW,Y,Z,W^2)	$(483,308,132,42;966)$	$x^2+w(y^3+z^7+w^{22})$
$(2,3,7,45)$	(X,YW,Z,W^3)	$(315,224,90,42;672)$	$y^3+w(x^2+z^7+w^{15})$
$(2,3,7,49)$	(X,Y,ZW,W^7)	$(147,98,48,42;336)$	$z^7+w(x^2+y^3+w^7)$
$(2,3,8,25)$	(XZ,Y,Z^2,W)	$(375,200,150,24;750)$	$x^2+z(y^3+z^4+w^{25})$
$(2,3,8,26)$	(XZW,Y,Z^2,W^2)	$(207,104,78,24;414)$	$x^2+zw(y^3+z^4+w^{13})$
$(2,3,9,19)$	(X,YZ,Z^3,W)	$(171,152,114,18;456)$	$y^3+z(x^2+z^3+w^{19})$
$(2,3,9,21)$	(X,YZW,Z^3,W^3)	$(63,62,42,18;186)$	$y^3+zw(x^2+z^3+w^7)$
$(2,3,10,16)$	(XZW,Y,Z^2,W^2)	$(159,80,48,30;318)$	$x^2+zw(y^3+z^5+w^8)$
$(2,3,13,13)$	(W^{13},X,Y,ZW)	$(78,39,26,12;156)$	$w^{13}+x(x+y^2+z^3)$
$(2,4,5,21)$	(XY,Y^2,Z,W)	$(315,210,84,20;630)$	$x^2+y(y^2+z^5+w^{21})$
$(2,4,5,22)$	(XYW,Y^2,Z,W^2)	$(175,110,44,20;350)$	$x^2+yw(y^2+z^5+w^{11})$
$(2,4,6,13)$	(XYZ,Y^2,Z^2,W)	$(143,78,52,12;286)$	$x^2+yz(y^2+z^3+w^{13})$
$(2,4,6,14)$	$(XYZW,Y^2,Z^2,W^2)$	$(83,42,28,12;166)$	$x^2+yzw(y^2+z^3+w^7)$
$(2,4,7,10)$	(XYW,Y^2,W^2,Z)	$(119,70,28,20;238)$	$x^2+yz(y^2+z^5+w^7)$
$(2,5,5,11)$	(Y^5,X,YZ,W)	$(110,55,44,10;220)$	$z^5+x(x+y^2+w^{11})$
$(2,5,5,15)$	(Y^5,X,YZW,W^5)	$(30,15,14,10;70)$	$z^5+xw(x+y^2+w^3)$
$(2,5,6,8)$	(XZW,Z^2,W^2,Y)	$(95,40,30,24;190)$	$x^2+yz(y^3+z^4+w^5)$
$(2,5,7,7)$	(W^7,X,ZW,Y)	$(70,35,20,14;140)$	$z^7+x(x+y^2+w^5)$
$(2,7,7,7)$	(Z^7,Y^7,X,YZW)	$(14,14,7,6;42)$	$w^7+xy(x+y+z^2)$
$(3,3,4,13)$	(X^3,XY,Z,W)	$(156,104,39,12;312)$	$y^3+x(x+z^4+w^{13})$
$(3,3,4,15)$	(X^3,XYW,Z,W^3)	$(60,44,15,12;132)$	$y^3+xw(x+z^4+w^5)$
$(3,3,5,8)$	(X^3,XY,Z,W)	$(120,80,24,15;240)$	$y^3+x(x+z^5+w^8)$
$(3,3,5,9)$	(X^3,XYW,W^3,Z)	$(45,35,15,9;105)$	$y^3+xz(x+z^3+w^5)$
$(3,3,6,7)$	(X^3,XYZ,Z^3,W)	$(42,35,21,6;105)$	$y^3+xz(x+z^2+w^7)$
$(3,3,6,9)$	$(X^3,XYZW,Z^3,W^3)$	$(18,17,9,6;51)$	$y^3+xzw(x+z^2+w^3)$
$(3,4,4,8)$	(Z^4,YZW,W^4,X)	$(24,15,12,8;60)$	$y^4+xz(x+z^2+w^3)$
$(3,4,5,5)$	(W^5,ZW,X,Y)	$(60,24,20,15;120)$	$y^5+x(x+z^3+w^4)$
$(3,5,5,5)$	(W^5,Z^5,YZW,X)	$(15,15,9,5;45)$	$z^5+xy(x+y+w^3)$
$(4,4,4,5)$	(Y^4,Z^4,XYZ,W)	$(20,20,15,4;60)$	$z^4+xy(x+y+w^5)$
$(4,4,4,8)$	$(X^4,Y^4,XYZW,W^4)$	$(8,8,7,4;28)$	$z^4+xyw(x+y+w^2)$
$(5,5,5,5)$	$(X^5,Y^5,Z^5,XYZW)$	$(5,5,5,4;20)$	$w^5+xyz(x+y+z)$

Table 1.2: Hypersurface generalized triangle singularities in dimension 3

2 Proof of Theorem 1.2

Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a signature such that R is a hypersurface, and

$$\overline{T} = \bigoplus_{\vec{v} \in L} \overline{T}_{\vec{v}} = \mathbb{C}\left[\overline{X}_1, \ldots, \overline{X}_n\right] \tag{2.1}$$

be a polynomial ring graded by the abelian group L in (1.12). The Veronese subring of $\overline{T}$ over $\mathbb{Z}\vec{\omega}$ is denoted by

$$\overline{R} = \bigoplus_{k \in \mathbb{Z}} \overline{R}_k, \qquad \overline{R}_k = \overline{T}_{k\vec{\omega}}. \tag{2.2}$$

We write

$$[i, j] = \{i, i+1, \ldots, j\} \tag{2.3}$$

for a pair (i, j) of integers with $i \le j$. Let $\varphi \colon \overline{T} \to T = \overline{T}/\left(\overline{F}\right)$ be the natural projection, where

$$\overline{F} = \sum_{i=1}^{n} \overline{Y}_i, \qquad \overline{Y}_i = \overline{X}_i^{p_i} \text{ for } i \in [1, n]. \tag{2.4}$$

The restriction $\varphi|_{\overline{R}} \colon \overline{R} \to R = \overline{R}/\left(\overline{F}\right)$ will also be denoted by φ by abuse of notation. We write $X_i = \varphi(\overline{X}_i)$ for $i \in [1, n]$. Any element $\vec{v} \in L$ can be written uniquely as

$$\vec{v} = \ell\vec{c} + \sum_{i=1}^{n} a_i \vec{X}_i, \tag{2.5}$$

where $\ell \in \mathbb{Z}$ and $0 \le a_i \le p_i - 1$ for any $i \in [1, n]$. This defines functions $\ell \colon L \to \mathbb{Z}$ and $a_i \colon L \to [0, p_i - 1]$ for $i \in [1, n]$. Any element of $\overline{T}_{\vec{v}}$ for $\vec{v} \in L$ can be written as the product $\overline{M}(\vec{v})\overline{P}$, where $\overline{M}(\vec{v})$ is the monomial defined by

$$\overline{M}(\vec{v}) = \prod_{i=1}^{n} \overline{X}_i^{a_i(\vec{v})} \tag{2.6}$$

and $\overline{P} \in \mathbb{C}\left[\overline{Y}_1, \ldots, \overline{Y}_n\right]_\ell$ is a homogeneous polynomial of degree ℓ.

Let $\mathfrak{m} = \bigoplus_{k=1}^{\infty} R_k$ be the irrelevant ideal of R. Since we assume that R is a hypersurface, that is, $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = n$, we can choose a set $\overline{\Xi} = \{\overline{x}_i\}_{i=1}^n \subset \overline{R}$ of monomials whose image $\Xi = \varphi\left(\overline{\Xi}\right) = \{x_i = \varphi\left(\overline{x}_i\right)\}_{i=1}^n$ forms a basis of the vector space $\mathfrak{m}/\mathfrak{m}^2$. By the following Lemma 2.1, the ring R is generated by Ξ over $\mathbb{C}$.

Lemma 2.1. *Let S be a subset of $\mathfrak{m}$ consisting of homogeneous elements, that is, $S \subset \bigcup_{k=1}^{\infty} R_k$. Then S generates the ring R over $\mathbb{C}$ if and only if the image of S spans the vector space $\mathfrak{m}/\mathfrak{m}^2$.*

Proof. Let $\mathbb{C}S$ denote the vector space spanned by S. If S generates R, that is, $\mathbb{C}[S] = R$, then $\mathfrak{m}/\mathfrak{m}^2 = (\mathbb{C}[S] \cap \mathfrak{m})/\mathfrak{m}^2 = (\mathbb{C}S + \mathfrak{m}^2)/\mathfrak{m}^2$. Hence the image of S spans $\mathfrak{m}/\mathfrak{m}^2$. Conversely, assume that the image of S spans $\mathfrak{m}/\mathfrak{m}^2$. In order to prove that S generates R by induction, suppose that $\mathbb{C}[S]$ contains R_k for all $k < n$. Then $\mathbb{C}[S] \supset \mathfrak{m}^2 \cap R_n$. Since the image of S spans $\mathfrak{m}/\mathfrak{m}^2$, we have $R_n = \mathbb{C}S \cap R_n + \mathfrak{m}^2 \cap R_n$, which implies that $\mathbb{C}[S]$ contains R_n. Therefore we obtain $\mathbb{C}[S] = R$ by induction. ∎

We also set

$$\begin{gathered}\nu = 1 - \sum_{i=1}^{n} \frac{1}{p_i}, \quad N = \operatorname{lcm}\{p_i \mid i \in [1,n]\}, \\ N_i = \operatorname{lcm}\{p_j \mid j \in [1,n] \setminus \{i\}\}, \quad q_i = \nu p_i N_i. \end{gathered} \tag{2.7}$$

Lemma 2.2. *For any $i \in [1,n]$ and any $m \in \mathbb{N}$, we have $\overline{X}_i^m \in \overline{R}$ if and only if $q_i \mid m$.*

Proof. Note that $\overline{R}_k$ contains a pure power of $\overline{X}_i$ if and only if

$$a_j(k\vec{\omega}) = 0 \qquad \text{for any } j \in [1,n] \setminus \{i\}, \tag{2.8}$$

where $a_j \colon L \to [0, p_j - 1]$ are defined by (2.5). Since

$$a_j(k\vec{\omega}) = p_j \left\lceil \frac{k}{p_j} \right\rceil - k \qquad \text{for any } j \in [1,n], \tag{2.9}$$

the condition (2.8) holds if and only if k is an integer multiple of N_i. It follows from

$$N_i\vec{\omega} = N_i\vec{c} - \sum_{j\in[1,n]\setminus\{i\}} \frac{N_i}{p_j}\vec{c} - N_i\vec{X}_i = \left(\nu + \frac{1}{p_i}\right) N_i\vec{c} - N_i\vec{X}_i = q_i\vec{X}_i \tag{2.10}$$

that the pure power of $\overline{X}_i$ contained in $\overline{R}_{N_i}$ is $\overline{X}_i^{q_i}$, and Lemma 2.2 is proved. □

Corollary 2.3. *If $\overline{X}_i^m \in \overline{\Xi}$ for some $i \in [1,n]$ and $m \in \mathbb{N}$, then one has $m = q_i$.*

Proof. This is immediate from Lemma 2.2. □

Lemma 2.4. *For any $k \in \mathbb{N}$ and any $\overline{Z} \in \overline{R}_k$, there exist $\overline{P} \in \mathbb{C}\left[\overline{Y}_1, \ldots, \overline{Y}_n\right]_{\ell(k\vec{\omega})-1}$ and $\overline{Q} \in \mathbb{C}\left[\overline{x}_1, \ldots, \overline{x}_n\right] \cap \overline{R}_k$ such that $\overline{Z} = \overline{M}(k\vec{\omega})\overline{P}\overline{F} + \overline{Q}$.*

Proof. Since Ξ generates R as a ring, there exist $\overline{P}' \in \overline{T}_{k\vec{\omega}-\vec{c}}$ and $\overline{Q} \in \mathbb{C}\left[\overline{x}_1, \ldots, \overline{x}_n\right]$ such that $\overline{Z} = \overline{P}'\overline{F} + \overline{Q}$. It follows from the definition of ℓ and $\overline{M}$ that $\ell(k\vec{\omega} - \vec{c}) = \ell(k\vec{\omega}) - 1$ and $\overline{M}(k\vec{\omega} - \vec{c}) = \overline{M}(k\vec{\omega})$. Hence there exists $\overline{P} \in \mathbb{C}\left[\overline{Y}_1, \ldots, \overline{Y}_n\right]_{\ell(k\vec{\omega})-1}$ such that $\overline{P}' = \overline{M}(k\vec{\omega})\overline{P}$. □

Lemma 2.5. *If $\overline{x}_i = \overline{X}_i^{q_i}$ for all $i \in [1,n]$, then we have $R = T$ and $\overline{x}_i = \overline{X}_i$ for all $i \in [1,n]$.*

Proof. For each i, fix a sufficiently large k with $k \equiv -1 \bmod p_i$. Since

$$k\vec{\omega} = k\vec{c} - k\sum_{i=1}^{n} \vec{X}_i, \tag{2.11}$$

we have $a_i(k\vec{\omega}) = 1$, so that there exists a monomial $\overline{G} \in \overline{T}_{k\vec{\omega}-\vec{X}_i}$ such that $\overline{X}_i\overline{G} \in \overline{R}_k$ and $\overline{X}_i \nmid \overline{G}$. By applying Lemma 2.4, we have

$$\overline{X}_i\overline{G} = \overline{M}(k\vec{\omega})\overline{P}\overline{F} + \overline{Q}, \quad \overline{X}_i \mid \overline{M}(k\vec{\omega}), \quad \text{and} \quad \overline{X}_i^2 \nmid \overline{M}(k\vec{\omega}). \tag{2.12}$$

Assume for contradiction that $q_i > 1$. By comparing terms of degree 1 in the variable X_i in (2.12), we obtain

$$\overline{X}_i\overline{G} = \overline{M}(k\vec{\omega}) \cdot \overline{P}|_{\overline{X}_i=0} \cdot \overline{F}|_{\overline{X}_i=0}. \tag{2.13}$$

Since $\overline{F}|_{\overline{X}_i=0} = \sum_{j\in[1,n]\setminus\{i\}} \overline{X}_j^{p_j}$ is not a monomial, the right hand side of (2.13) is not a monomial. This contradicts the fact that the left hand side is a monomial, and Lemma 2.5 is proved. □

Lemma 2.6. *If there exist $i, j \in [1, n]$ such that $i \neq j$, $\overline{X}_i^{q_i} \notin \overline{\Xi}$, and $\overline{X}_i^a \overline{X}_j^b \notin \overline{\Xi}$ for all $a, b \geq 1$, then $\overline{X}_j^{q_j} \in \overline{\Xi}$ and $p_i \mid q_i$.*

Proof. By Lemma 2.2, we have $\overline{X}_i^{q_i} \in R$. Hence we have

$$\overline{X}_i^{q_i} = \overline{M}\left(q_i \vec{X}_i\right) \overline{PF} + \overline{Q} \tag{2.14}$$

by Lemma 2.4. Let $\pi \colon \mathbb{C}\left[\overline{X}_1, \ldots, \overline{X}_n\right] \to \mathbb{C}\left[\overline{X}_i, \overline{X}_j\right]$ be the surjective ring homomorphism defined by

$$\pi\left(\overline{X}_k\right) = \begin{cases} \overline{X}_k & k = i, j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.15}$$

By projecting (2.14) by π, we obtain

$$\overline{X}_i^{q_i} = \pi\left(M\left(q_i \vec{X}_i\right) \overline{P}\right) \cdot \left(\overline{X}_i^{p_i} + \overline{X}_j^{p_j}\right) + \pi\left(\overline{Q}\right). \tag{2.16}$$

It follows from the assumption of Lemma 2.6 that the only element in $\overline{\Xi}$ whose image by π does not vanish is a polynomial in $\overline{X}_j$. Hence we have $\pi\left(\overline{Q}\right) \in \mathbb{C}\left[\overline{X}_j\right]$. If $\pi\left(\overline{Q}\right) = 0$, then the right hand side of (2.16) is not a monomial, which contradicts the fact that the left hand side is a monomial. Hence we have $\pi\left(\overline{Q}\right) \neq 0$, so that $\overline{X}_j^m \in \overline{\Xi}$ for some $m \in \mathbb{N}$. This implies $m = q_j$ by Corollary 2.3. It follows from (2.16) that $\overline{X}_i^{q_i} - \pi\left(\overline{Q}\right)$ is divisible by $\pi\left(\overline{M}(q_i \vec{X}_i)\right)$. Together with the fact that $\pi\left(\overline{Q}\right) \in \mathbb{C}\left[\overline{X}_j\right]$, this implies that $\overline{M}\left(q_i \vec{X}_i\right) = 1$. Hence p_i divides q_i, and Lemma 2.6 is proved. $\square$

Lemma 2.7. *Let i, j, k be distinct elements of $[1, n]$. If $\overline{X}_i^{q_i}, \overline{X}_j^{q_j} \notin \overline{\Xi}$ and $\overline{X}_k^{q_k} \in \overline{\Xi}$, then there exists an element in $\overline{\Xi}$ of the form $\overline{X}_i^a \overline{X}_j^b \overline{X}_k^c$ with $(a, b) \neq (0, 0)$ and $c \geq 1$.*

Proof. Assume for contradiction that $\overline{X}_i^a \overline{X}_j^b \overline{X}_k^c \notin \overline{\Xi}$ for all (a, b, c) with $(a, b) \neq (0, 0)$ and $c \geq 1$. In particular, we have $\overline{X}_i^a \overline{X}_k^c \notin \overline{\Xi}$ for all $a, c \geq 1$. Together with the assumption that $\overline{X}_i^{q_i} \notin \overline{\Xi}$, this implies $p_i \mid q_i$ by Lemma 2.6. If we set $q = q_i / p_i$, then we have

$$\overline{Y}_i^q = \overline{PF} + \overline{Q} \tag{2.17}$$

by Lemma 2.4, since $\overline{Y}_i^q \in T_{q\vec{c}}$ and $\overline{M}(q\vec{c}) = 1$. Let $\pi \colon \mathbb{C}\left[\overline{X}_1, \ldots, \overline{X}_n\right] \to \mathbb{C}\left[\overline{X}_i, \overline{X}_j, \overline{X}_k\right]$ be the surjective homomorphism defined by

$$\pi(\overline{X}_l) = \begin{cases} \overline{X}_l & l = i, j, k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

We can write

$$\pi\left(\overline{P}\right) = \overline{Y}_i^{q-1} + \overline{A}_1 \overline{Y}_i^{q-2} + \cdots + \overline{A}_{q-1}, \tag{2.19}$$

where $\overline{A}_l \in \mathbb{C}\left[\overline{Y}_j, \overline{Y}_k\right]$ for $l \in [1, q-1]$. Then we have

$$\begin{aligned} \pi\left(\overline{PF}\right) &= \pi\left(\overline{P}\right) \pi\left(\overline{F}\right) \\ &= \left(\overline{Y}_i^{q-1} + \overline{A}_1 \overline{Y}_i^{q-2} + \cdots + \overline{A}_{q-1}\right) \left(\overline{Y}_i + \overline{Y}_j + \overline{Y}_k\right) \\ &= \overline{Y}_i^q + \left(\overline{A}_1 + \overline{Y}_j + \overline{Y}_k\right) \overline{Y}_i^{q-1} + \left(\overline{A}_2 + \left(\overline{Y}_j + \overline{Y}_k\right) \overline{A}_1\right) \overline{Y}_i^{q-2} + \cdots + \left(\overline{Y}_j + \overline{Y}_k\right) \overline{A}_{q-1}. \end{aligned}$$

We shall show that $\overline{X}_j \mid \overline{A}_l$ for all $l \in [1, q-1]$. It follows from the assumption that every monomial appearing in $\pi(\overline{Q})$ is either divisible by $\overline{X}_i\overline{X}_j$ or consists only of $\overline{X}_k$. Since all monomials appearing in $\left(\overline{Y}_j + \overline{Y}_k\right)\overline{A}_{q-1}$ are not divisible by $\overline{X}_i$, they must be in $\mathbb{C}[\overline{X}_k]$. This implies that $\overline{A}_{q-1} = 0$. Since all monomials appearing in $\left(\overline{A}_{q-1} + \left(\overline{Y}_j + \overline{Y}_k\right)\overline{A}_{q-2}\right)\overline{Y}_i$ contains $\overline{X}_i$, they must be divisible by $\overline{X}_i\overline{X}_j$. Hence we must have $\overline{X}_j \mid \overline{A}_{q-2}$. By repeating the same argument, we obtain $\overline{X}_j \mid \overline{A}_l$ for all $l \in [1, q-1]$. In particular, one has $\overline{X}_j \mid \overline{A}_1$. It follows that the monomial $\overline{Y}_i^{q-1}\overline{Y}_k$ from $(\overline{A}_1 + \overline{Y}_j + \overline{Y}_k)\overline{Y}_i^{q-1}$ do not cancel with any other terms. Since this monomial is neither divisible by $\overline{X}_i\overline{X}_j$ nor consists only of $\overline{X}_k$, this is a contradiction, and Lemma 2.7 is proved. ☐

The following lemma is the key to proving Theorem 1.2. Set

$$I := \left\{ i \in [1, n] \;\middle|\; \overline{X}_i^{q_i} \in \Xi \right\}. \tag{2.20}$$

Lemma 2.8. *If $n \geq 4$, then we have $|I| \geq n - 1$.*

Proof. Assume for contradiction that $n \geq 4$ and $r := n - |I| \geq 2$. If $i \neq j$ and $i, j \notin I$, then we have $\overline{X}_i^a\overline{X}_j^b \in \Xi$ for some $a, b \geq 1$ by Lemma 2.6. It follows that

$$\# \left\{ \overline{X}_i^a\overline{X}_j^b \in \Xi \;\middle|\; i, j \notin I,\ a, b \geq 1 \right\} \geq \binom{r}{2}. \tag{2.21}$$

Similarly, Lemma 2.7 implies that

$$\# \left\{ \overline{X}_i^a\overline{X}_j^b\overline{X}_k^c \in \Xi \;\middle|\; i, j \notin I,\ k \in I,\ (a, b) \neq (0, 0),\ c \geq 1 \right\} \geq |I| = n - r. \tag{2.22}$$

It follows from (2.20)–(2.22) that

$$\#\Xi = n \geq |I| + \binom{r}{2} + |I| = (n - r) + \frac{1}{2}r(r - 1) + (n - r), \tag{2.23}$$

and hence

$$n \leq -\frac{1}{2}r(r - 5). \tag{2.24}$$

Since

$$\max \left\{ -\frac{1}{2}r(r - 5) \;\middle|\; r \in [0, n] \right\} = 3, \tag{2.25}$$

the inequality (2.24) contradicts the assumption $n \geq 4$, and Lemma 2.8 is proved. ☐

Proposition 2.9. *If $n \geq 4$, we have $\nu = 1/N$.*

Proof. If $R = T$, we have $q_i = 1$ for all $i \in [1, n]$ by Lemma 2.2. Hence

$$q_i = \nu p_i N_i = \nu \cdot \operatorname{lcm}(p_i, N_i) \cdot \gcd(p_i, N_i) = \nu N \cdot \gcd(p_i, N_i) = 1. \tag{2.26}$$

Since νN is an integer, it follows that

$$\nu N = \gcd(p_i, N_i) = 1. \tag{2.27}$$

If $R \subsetneq T$, then we have $\overline{X}_i^{q_i} \notin \Xi$ for some $i \in [1,n]$ by Lemma 2.5. Since we have $\overline{X}_l^{q_l} \in \Xi$ for all $l \in [1,n] \setminus \{i\}$ by Lemma 2.8, we can set $\overline{x}_l = \overline{X}_l^{q_l}$ for $l \in [1,n] \setminus \{i\}$. Then we have $\overline{x}_i \neq \overline{X}_i^{q_i}$. Let m be an element of $[1,n] \setminus \{i\}$ such that the variable $\overline{X}_m$ appears in $\overline{x}_i$, and fix any distinct elements j and k of $[1,n] \setminus \{i,m\}$. Then it follows that $\overline{X}_i^a \overline{X}_j^b \notin \Xi$ for all $a, b \geq 1$. This implies that $p_i \mid q_i$ by Lemma 2.6.

Let $\pi \colon \mathbb{C}\left[\overline{X}_1, \ldots, \overline{X}_n\right] \to \mathbb{C}\left[\overline{X}_i, \overline{X}_j, \overline{X}_k\right]$ be the surjective homomorphism defined by

$$\pi\left(\overline{X}_l\right) = \begin{cases} \overline{X}_l & l = i, j, k, \\ 0 & \text{otherwise}, \end{cases} \tag{2.28}$$

just as in the proof of Lemma 2.7. It follows from the choice of j and k that

$$\pi\left(\overline{x}_l\right) = 0 \text{ for any } l \in [1,n] \setminus \{j,k\}. \tag{2.29}$$

If we write $q = q_i/p_i$, then the same argument as in the proof of Lemma 2.7 shows

$$\overline{Y}_i^q = \overline{PF} + \overline{Q} \tag{2.30}$$

and

$$\pi(\overline{PF}) = \overline{Y}_i^q + \left(\overline{A}_1 + \overline{Y}_j + \overline{Y}_k\right)\overline{Y}_i^{q-1} + \left(\overline{A}_2 + \left(\overline{Y}_j + \overline{Y}_k\right)\overline{A}_1\right)\overline{Y}_i^{q-2} + \cdots + \left(\overline{Y}_j + \overline{Y}_k\right)\overline{A}_{q-1}. \tag{2.31}$$

If follows from (2.29) that

$$\pi\left(\overline{Q}\right) \in \mathbb{C}\left[\overline{X}_j^{q_j}, \overline{X}_k^{q_k}\right]. \tag{2.32}$$

By projecting (2.30) by π, we obtain

$$\pi\left(\overline{Q}\right) = \overline{Y}_i^q - \pi\left(\overline{PF}\right), \tag{2.33}$$

which together with (2.31) and (2.32) gives

$$\overline{A}_l = (-\overline{Y}_j - \overline{Y}_k)^l \quad \text{for any } l \in [1, q-1] \tag{2.34}$$

and

$$\pi(Q) = (-\overline{Y}_j - \overline{Y}_k)^q. \tag{2.35}$$

It follows from (2.32) and (2.35) that $\overline{Y}_j^a \overline{Y}_k^{q-a} \in \mathbb{C}\left[\overline{X}_j^{q_j}, \overline{X}_k^{q_k}\right]$ for any $a \in [0,q]$. Since $\overline{Y}_j^a \overline{Y}_k^{q-a} = \overline{X}_j^{ap_j} \overline{X}_k^{(q-a)p_k}$, this implies $q_j \mid p_j$ and $q_k \mid p_k$. Hence both $p_j/q_j = 1/\nu N_j$ and $p_k/q_k = 1/\nu N_k$ are integers. The product $u := \nu N$ is an integer by the definitions of ν and N in (2.7). The definitions of N_j, N_k and N in (2.7) and the fact that both $1/\nu N_j = N/u N_j$ and $1/\nu N_k = N/u N_k$ are integers imply $u = 1$, and Proposition 2.9 is proved. □

Corollary 2.10. *We have $q_i = \gcd(p_i, N_i)$ and, in particular, $q_i \mid p_i$ for any $i \in [1,n]$.*

Proof. It follows from (2.7) and $\nu = 1/N$ that

$$q_i = \nu p_i N_i = \nu \cdot \operatorname{lcm}(p_i, N_i) \cdot \gcd(p_i, N_i) = \nu N \cdot \gcd(p_i, N_i) = \gcd(p_i, N_i). \tag{2.36}$$

□

Corollary 2.11. *We have either*

(1) $R = T$ *and* $x_i = X_i$ *for all* $i \in [1, n]$*, or*

(2) $R \neq T$ *and there exists* $i \in [1, n]$ *such that* $q_i = p_i$ *and* $x_j = \begin{cases} X_j^{q_j} & j \neq i, \\ \prod_{q_k \neq 1} X_k & j = i. \end{cases}$

Proof. It follows from (2.7) and Proposition 2.9 that

$$\frac{1}{N} = 1 - \sum_{i=1}^{n} \frac{1}{p_i}. \tag{2.37}$$

Hence we have

$$\begin{aligned} (N-1)\vec{\omega} &= (N-1)\vec{c} - (N-1)\sum_{i=1}^{n} \vec{X}_i & (2.38)\\ &= \left\{ (N-1) - \sum_{i=1}^{n} \frac{N}{p_i} \right\} \vec{c} + \sum_{i=1}^{n} \vec{X}_i & (2.39)\\ &= \sum_{i=1}^{n} \vec{X}_i, & (2.40) \end{aligned}$$

so that $\prod_{i=1}^{n} X_i \in R_{N-1}$. If $R = T$, then we can set $x_i = X_i$ for all $i \in [1, n]$. If $R \neq T$, then we have $|I| = n - 1$ by Lemma 2.8, and there exists $i \in [1, n]$ such that $x_j = X_j^{q_j}$ for $j \in [1, n] \setminus \{i\}$. Note that we have $\overline{F} \in \overline{T}_{\vec{c}}$ and $\vec{c} = N\vec{\omega}$. We can remove X_k such that $q_k = 1$ from $\prod_{k=1}^{n} X_k$ to obtain an element $\prod_{q_k \neq 1} X_k$, which is one of the generators of R. Hence $x_i = \prod_{q_k \neq 1} X_k$. Since we have $X_i^{q_i} \in R$ (Lemma 2.2) and $q_i \mid p_i$ (Corollary 2.10), it follows that $q_i = p_i$. □

Lemma 2.12. *For any positive integer* n*, there exist only finitely many sequences* $(p_1, \ldots, p_n)$ *of* n *positive integers satisfying*

$$\sum_{i=1}^{n} \frac{1}{p_i} + \frac{1}{N} = 1, \tag{2.41}$$

where $N = \operatorname{lcm}\{p_i \mid i \in [1, n]\}$.

Proof. We may assume $p_1 \leq p_2 \leq \cdots \leq p_n$. Then we have $p_1 \leq N$ and

$$\frac{n+1}{p_1} \geq \sum_{i=1}^{n} \frac{1}{p_i} + \frac{1}{N} = 1, \tag{2.42}$$

so that

$$p_1 \leq n + 1. \tag{2.43}$$

Hence there are only finitely many possibilities for p_1. If we fix p_1, then we have

$$\frac{n}{p_2} \geq \sum_{i=2}^{n} \frac{1}{p_i} + \frac{1}{N} = 1 - \frac{1}{p_1} \tag{2.44}$$

since $p_2 \leq N$. This leaves only finitely many possibilities for p_2. By repeating the same argument, we can see that there are only finitely many possibilities for $(p_1, \ldots, p_n)$. □

Proof of Theorem 1.2. For any $n \geq 4$, Proposition 2.9 and Lemma 2.12 give a finite list of possible signatures for hypersurface generalized triangle singularities, and Theorem 1.2 is proved. $\square$

For each signature $\boldsymbol{p} = (p_1, \dots, p_n)$ with $\nu = 1/N$, we can check if R is a hypersurface as follows: If $q_i = 1$ for any $i \in [1, n]$, then we have $R = T$ and R is a hypersurface.

If R is a hypersurface and $R \neq T$, then there exists $i \in [1, n]$ such that $q_i = p_i$ and

$$x_j = \begin{cases} X_j^{q_j} & j \neq i, \\ \prod_{q_k \neq 1} X_k & j = i \end{cases} \tag{2.45}$$

by Corollary 2.11. Assume that there exists $i \in [1, n]$ such that $q_i = p_i$. Fix any such i and define $\{x_j\}_{j=1}^n$ by (2.45). Let R' be the subring of T generated by $\{x_j\}_{j=1}^n$, so that R is a hypersurface if and only if $R = R'$. It follows from $\nu = 1/N$ that $q_j \mid p_j$ for any $j \in [1, n]$ (cf. Corollary 2.10). Hence $Y_j := X_j^{p_j}$ is contained in R' for any $j \in [1, n]$. Note that any element of $T_{\vec{v}}$ for $\vec{v} \in L$ can be written as the product $M(\vec{v})P$, where $M(\vec{v}) := \varphi(\overline{M}(\vec{v}))$ is the image of $\overline{M}(\vec{v})$ defined by (2.6), and P is a homogeneous element of $\mathbb{C}[Y_1, \dots, Y_n]/(Y_1 + \cdots + Y_n)$. Since $M((k + N)\vec{\omega}) = M(k\vec{\omega})$ for any $k \in \mathbb{Z}$, the ring R is generated by $\{Y_j\}_{j=1}^n$ and $\{M(k\vec{\omega})\}_{k=0}^{N-1}$. Therefore we have $R = R'$ if and only if $M(k\vec{\omega}) \in R'$ for $0 \leq k \leq N - 1$. Table 1.2 is obtained in this way.

3 Proof of Theorem 1.3

We keep the same notations as in Section 2. Given a signature $\boldsymbol{p} = (p_1, \dots, p_n)$, we define a group $G \subset \mathrm{GL}_n(\mathbb{C})$ by

$$G = \left\{ \mathrm{diag}(\alpha_1, \dots, \alpha_n) \;\middle|\; \alpha_1^{p_1} = \cdots = \alpha_n^{p_n} = \prod_{i=1}^n \alpha_i = 1 \right\}. \tag{3.1}$$

The group G acts naturally on T in such a way that $\mathrm{diag}(\alpha_1, \dots, \alpha_n) \in G$ maps $X_i \in T$ to $\alpha_i X_i$ for $i \in [1, n]$.

Lemma 3.1. *If $n \geq 4$ and R is a hypersurface, then R coincides with the invariant ring T^G.*

Proof. We have $N\vec{\omega} = N\nu\vec{c}$, which is equal to $\vec{c}$ by Proposition 2.9. This shows that $\mathbb{Z}\vec{\omega} \supset \mathbb{Z}\vec{c}$. Note that

$$L/\mathbb{Z}\vec{c} \cong \bigoplus_{i=1}^n \mathbb{Z}\vec{X}_i/(p_i\vec{X}_i) \tag{3.2}$$

and

$$\vec{\omega} \equiv -\sum_{i=1}^n \vec{X}_i \mod \vec{c}. \tag{3.3}$$

It follows that

$$L/\mathbb{Z}\vec{\omega} \cong \left(\bigoplus_{i=1}^n \mathbb{Z}\vec{X}_i \right) \Bigg/ \left(p_i\vec{X}_i, \sum_{i=1}^n \vec{X}_i \right). \tag{3.4}$$

This allows us to identify $L/\mathbb{Z}\vec{\omega}$ with the group of characters of G, so that the ring R, which is the Veronese subring over $\vec{\omega}$, is exactly the G-invariant part of T. $\square$

Proposition 3.2. *If $n \geq 4$ and R is a hypersurface, then R has an isolated singularity if and only if $R = T$.*

Proof. The 'if' part is clear since T has an isolated singularity. To prove the 'only if' part, assume that $R \subsetneq T$. Then we have $s := \gcd(p_i, p_j) \neq 1$ for some $1 \leq i < j \leq n$ by Lemma 2.2 and Corollary 2.10. Define a subset of

$$\operatorname{Spec} T = \{(X_1, \ldots, X_n) \in \mathbb{A}^n \mid X_1^{p_1} + \cdots + X_n^{p_n} = 0\} \tag{3.5}$$

by

$$P = \{(X_1, \ldots, X_n) \in \operatorname{Spec} T \mid X_i = X_j = 0 \text{ and } X_k \neq 0 \text{ for any } k \neq i, j\}. \tag{3.6}$$

Then the stabilizer subgroup of any point in P with respect to the action of G is given by

$$\{\operatorname{diag}(\alpha_1, \ldots, \alpha_n) \mid \alpha_i^s = 1,\ \alpha_i \alpha_j = 1 \text{ and } \alpha_k = 1 \text{ for any } k \neq i, j\}. \tag{3.7}$$

This is isomorphic to a cyclic subgroup of $\mathrm{SL}_2(\mathbb{C})$, so that $\operatorname{Spec} R = (\operatorname{Spec} T)/G$ has a non-isolated family of A_{s-1}-singularities along P/G. □

Now we prove Theorem 1.3:

Proof of Theorem 1.3. To prove the 'if' part, assume that we have (1.15). Then we have

$$\nu := 1 - \sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{\prod_{i=1}^{n} p_i}. \tag{3.8}$$

It follows that for any $i \in [1, n]$, we have

$$\begin{aligned} q_i &:= \nu p_i N_i && (3.9)\\ &= \frac{1}{\prod_{i=1}^{n} p_i} \cdot p_i \cdot \operatorname{lcm}\{p_j \mid j \in [1, n] \setminus \{i\}\} && (3.10)\\ &\leq \frac{1}{\prod_{i=1}^{n} p_i} \cdot p_i \cdot \prod_{j \in [1, n] \setminus \{i\}} p_j && (3.11)\\ &= 1. && (3.12) \end{aligned}$$

This implies $q_i = 1$ since q_i is a positive integer by definition. Hence we have $R = T$ (Lemma 2.2), which clearly has an isolated singularity at the origin.

To prove the 'only if' part, assume that R has an isolated hypersurface singularity. Then Proposition 3.2 shows $R = T$, which implies $q_i = 1$ for any $i \in [1, n]$ by Lemma 2.2. Then one has $\nu = 1/N$ and $\gcd(p_i, N_i) = 1$ for any $i \in [1, n]$ by (2.27), which implies $N = \operatorname{lcm}\{p_i \mid i \in [1, n]\} = \prod_{i=1}^{n} p_i$ and (1.15) by (2.7). □

4 Proof of Theorem 1.4

Since we assume that $n \geq 4$ and R is a hypersurface, we have

$$\nu := 1 - \sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{N} \tag{4.1}$$

by Proposition 2.9. Define a function $m\colon \mathbb{Z} \to \mathbb{Z}$ by

$$m(k) := \ell(k\vec{\omega}) = k - \sum_{i=1}^{n} \left\lceil \frac{k}{p_i} \right\rceil, \tag{4.2}$$

where the function $\ell\colon L \to \mathbb{Z}$ is defined by (2.5). Recall that we have

$$R_k = \overline{M}(k\vec{\omega}) \cdot \left(\mathbb{C}[\overline{Y}_1, \dots, \overline{Y}_n] / \left(\overline{Y}_1 + \cdots + \overline{Y}_n\right)\right)_{m(k)}, \tag{4.3}$$

where $\overline{M}(k\vec{\omega})$ is defined by (2.6). Therefore the Hilbert series of R is given by

$$F(t) := \sum_{k=0}^{\infty} (\dim_{\mathbb{C}} R_k)\, t^k = \sum_{k=0}^{\infty} c(k) t^n, \tag{4.4}$$

where

$$c(k) = \begin{cases} \dbinom{m(k) + n - 2}{n - 2} & m(k) \geq 0, \\ 0 & m(k) < 0. \end{cases} \tag{4.5}$$

We can write

$$F(t) = \sum_{j=0}^{N-1} g_j(t) t^j, \tag{4.6}$$

where

$$g_j(t) = \sum_{k=0}^{\infty} c(j + Nk) t^{Nk}. \tag{4.7}$$

It follows from (4.1) and (4.2) that

$$m(j + Nk) = m(j) + \nu N k = m(j) + k. \tag{4.8}$$

Hence we have

$$c(j + Nk) = \begin{cases} \dbinom{m(j) + k + n - 2}{n - 2} & m(j) + k \geq 0, \\ 0 & m(j) + k < 0. \end{cases} \tag{4.9}$$

Therefore, by the following Lemma 4.1,

$$g_j(t) = \sum_{k=0}^{\infty} \binom{k + n - 2}{n - 2} t^{N(k - m(j))}. \tag{4.10}$$

Lemma 4.1. *We have $m(0) = 0$, $m(1) = -(n-1)$, and*

$$-(n-2) \leq m(k) \leq 0 \tag{4.11}$$

for $2 \leq k \leq N - 1$.

Proof. It is clear that $m(0) = 0$ and $m(1) = -(n-1)$. For $k \le N-1$, we have

$$m(k) \le k - \sum_{i=1}^{n} \frac{k}{p_i} = \nu k = \frac{k}{N} < 1,$$

so that $m(k) \le 0$. On the other hand, for $k \ge 2$, we have

$$m(k) \ge k - \sum_{i=1}^{n} \left(\frac{k}{p_i} + \frac{p_i - 1}{p_i} \right) = \nu k - n - (\nu - 1) = \nu(k-1) - (n-1) > -(n-1),$$

so that $m(k) \ge -(n-2)$. $\square$

Lemma 4.2. $\left(1 - t^N\right)^{n-1} F(t)$ *is a polynomial of degree* $(n-1)N + 1$.

Proof. By (4.10), we have

$$\left(1 - t^N\right)^{n-1} F(t) = \left(1 - t^N\right)^{n-1} \sum_{j=0}^{N-1} g_j(t) t^j \tag{4.12}$$

$$= \left(1 - t^N\right)^{n-1} \sum_{j=0}^{N-1} t^j \sum_{k=0}^{\infty} \binom{k+n-2}{n-2} t^{N(k-m(j))} \tag{4.13}$$

$$= \sum_{j=0}^{N-1} t^{j - N \cdot m(j)} \left(1 - t^N\right)^{n-1} \sum_{k=0}^{\infty} \binom{k+n-2}{n-2} t^{Nk} \tag{4.14}$$

$$= \sum_{j=0}^{N-1} t^{j - N \cdot m(j)}, \tag{4.15}$$

which is a polynomial of degree $(n-1)N + 1$ by Lemma 4.1. $\square$

Now we prove Theorem 1.4:

Proof of Theorem 1.4. If R is generated by elements of degrees $a_1, \ldots, a_n$ with one relation of degree h, then the Hilbert series of R is given by

$$F(t) = \frac{1 - t^h}{\prod_{i=1}^{n} (1 - t^{a_i})}. \tag{4.16}$$

By Lemma 4.2, we have

$$h = \sum_{i=1}^{n} a_i + 1. \tag{4.17}$$

This concludes the proof of Theorem 1.4. $\square$

Remark 4.3. By (4.16), we have

$$\lim_{t \to 1} (1-t)^{n-1} F(t) = \lim_{t \to 1} \frac{1 + t + \cdots + t^{h-1}}{\prod_{i=1}^{n} (1 + t + \cdots + t^{a_i - 1})} = \frac{h}{\prod_{i=1}^{n} a_i}. \tag{4.18}$$

Similarly, by (4.15), we have

$$\lim_{t\to 1}(1-t)^{n-1}F(t) = \lim_{t\to 1}\frac{(1-t)^{n-1}}{(1-t^N)^{n-1}}\sum_{j=0}^{N-1} t^{j-N\cdot m(j)} \tag{4.19}$$

$$= \frac{1}{N^{n-1}}\cdot N \tag{4.20}$$

$$= \frac{1}{N^{n-2}}. \tag{4.21}$$

Hence the following equality holds:

$$\frac{h}{\prod_{i=1}^{n} a_i} = \frac{1}{N^{n-2}}. \tag{4.22}$$

A related discussion can be found in [Wag80, Theorem 2.6].

References

[Arn75] V. I. Arnol′d, *Critical points of smooth functions*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 19–39. MR 0431217 (55 #4218)

[Dol75] I. V. Dolgačev, *Automorphic forms and quasihomogeneous singularities*, Funkcional. Anal. i Priložen. **9** (1975), no. 2, 67–68. MR 0568895 (58 #27958)

[Dol83] Igor Dolgachev, *Integral quadratic forms: applications to algebraic geometry (after V. Nikulin)*, Bourbaki seminar, Vol. 1982/83, Astérisque, vol. 105, Soc. Math. France, Paris, 1983, pp. 251–278. MR 728992 (85f:14036)

[Gab74] A. M. Gabrièlov, *Dynkin diagrams of unimodal singularities*, Funkcional. Anal. i Priložen. **8** (1974), no. 3, 1–6. MR 0367274 (51 #3516)

[GL87] Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297. MR MR915180 (89b:14049)

[Mil75] John Milnor, *On the 3-dimensional Brieskorn manifolds $M(p,q,r)$*, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), Princeton Univ. Press, Princeton, N. J., 1975, pp. 175–225. Ann. of Math. Studies, No. 84. MR 0418127 (54 #6169)

[Nik79] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238. MR 525944 (80j:10031)

[Pin77] Henry Pinkham, *Singularités exceptionnelles, la dualité étrange d'Arnold et les surfaces $K-3$*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 11, A615–A618. MR 0429876 (55 #2886)

[Wag80] Philip Wagreich, *Algebras of automorphic forms with few generators*, Trans. Amer. Math. Soc. **262** (1980), no. 2, 367–389. MR 586722 (82e:10044)

Kenji Hashimoto
School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul, 130-722, Korea
e-mail address : hashimoto@kias.re.kr

Hwayoung Lee
Department of Mathematical Sciences, Research Institute of Mathematics, Seoul National University, Gwanak-ro 1, Gwanak-gu, Seoul 151-747, Korea
e-mail address : hlee014@snu.ac.kr

Kazushi Ueda
Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.
e-mail address : kazushi@math.sci.osaka-u.ac.jp